

Multiple Scattering Formulation of Two Dimensional Acoustic and Electromagnetic Metamaterials

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Abstract. This work presents a multiple scattering formulation of two dimensional acoustic metamaterials. It is shown that in the low frequency limit multiple scattering allows us to define frequency-dependent effective acoustic parameters for arrays of both ordered and disordered cylinders. This formulation can lead to both positive and negative acoustic parameters, where the acoustic parameters are the scalar bulk modulus and the tensorial mass density and, therefore, anisotropic wave propagation is allowed with both positive or negative refraction index. It is also shown that the surface fields on the scatterer are the main responsible of the anomalous behavior of the effective medium, therefore complex scatterers can be used to engineer the frequency response of the effective medium, and some examples of application to different scatterers are given. Finally, the theory is extended to electromagnetic wave propagation, where Mie resonances are found to be the responsible of the metamaterial behavior. As an application, it is shown that it is possible to obtain metamaterials with negative permeability and permittivity tensors by arrays of all-dielectric cylinders and that anisotropic cylinders can tune the frequency response of these resonances.

1. Introduction

Fluid-like metamaterials or metafluids with negative constitutive parameters offer a new insight into acoustic wave propagation. Single negative metamaterials (SNM), in which either the mass density or the bulk modulus is negative [1, 2, 3], can be used, for example, designing of surface-like acoustic lens to overcome the diffraction limit [4, 5] or for the design of acoustic panels [6]. Double negative metamaterials (DNM)[7, 8, 9] present negative refraction [10, 11, 12, 13] and, as it is well known from electromagnetic wave theory, they can also be used to increase the resolution of conventional lens [14, 15, 16]. In general, anisotropic fluid-like metamaterials with acoustic parameters both positive and negative are necessary in the field of transformation acoustics for the design of several types of acoustic devices based on sound propagation [17, 18, 19, 20, 21].

The existence of frequency ranges in which the effective medium presents negative constitutive parameters is related with the subwavelength resonances of the individual scatterers that constitute the metamaterial, being these due to soft-scatterer resonances [7, 22, 23] or due to Helmholtz-like resonances [24, 25, 26, 8, 2]. The same phenomenon is found in electromagnetic waves under the name of Mie resonances [27, 28, 29, 30], and they present an alternate way of design electromagnetic metamaterials to that offered by split ring resonators [31] or metallodielectric composites [13], which have been the dominant structures so far. Therefore, metamaterials based on the resonances of the individual scatterers are important not only for acoustic but also for electromagnetic metamaterials.

It is known that a monopolar resonance in the scatterer is the responsible of the negative bulk modulus, and that a dipolar one is the responsible of the negative mass density [7]. However, the full effect of the ensemble of scatterers that constitute the effective medium has been partially explained only, as multiple scattering effects or anisotropic lattices have not been considered yet.

In this work, we give a comprehensive description of multiple of acoustic metamaterials by using a multiple scattering formulation. It is based on our previous results on homogenization of sonic crystals [32, 33, 34]. This formulation describes, in the long wavelength limit, an ensemble of orderer or disordered scatterers as an effective medium with frequency-dependent acoustic parameters, which are shown to be negative in certain frequency regions. The frequency-dependent parameters are given in terms of both the lattice symmetry and the surface fields on the scatterers, showing that these fields are the quantities that we have to manage in order to engineer the frequency response of our effective medium.

Therefore this formulation allow us, from one side, summarize all the previous results regarding SNM and DNM, from the other side, extend the theory to any type of radially symmetric scatterer and to non symmetric lattices, like rectangular arrangements in place of square or hexagonal ones, which are the more usually studied. Also, we show that the regions in which the metamaterial presents divergent or negative acoustic parameters is mainly a function of the surface fields of the scatterers, opening

therefore the possibility of engineer the scatterer in order to increase the frequency region in which the anomalous behavior occurs.

The paper is organized as follows: In section 2 the concept of quasi-static resonance is introduced, showing how these resonances lead to scatterers with locally negative parameters. Section Appendix A analyzes several examples of scatterers with locally negative parameters, showing that complex scatterers like fluid-like shells or anisotropic fluid-like cylinders can tune the metamaterial behavior of the scatterer. After that, in Section 4 the multiple scattering formulation is presented, analyzing first the case in which the scatterers have small radius (low filling fraction) and, later, the more general case, showing how anisotropy appears. In this section the theory is extended to include multipolar effects, but we see that they are not very important in principle. Finally, the application of the theory to electromagnetic waves is explained in section 5. The paper ends with a summary section.

2. Quasi-Static Resonances and Locally Negative Parameters

Homogenization theories for assemblies of scatterers are based on the low wavenumber (long wavelength) expansions of the fields in both the background and the scatterers. When working with metamaterials we assume that the wavenumber in the background is asymptotically small though we let the wavenumber inside the scatterer still be finite. Physically it means that outside the scatterers the wave field propagates through an effective medium but it is still allowed to the scatterers to have complex internal scattering processes, which will lead to locally (i.e., in a narrow frequency region) negative parameters, as will be explained in the following sections.

The simpler example of these scatterers, and the first studied in the next section, is the homogeneous fluid-like scatterer. If the speed of sound inside this scatterer is much smaller than that of the background, $c_a \ll c_b$, we will have that, for a given frequency ω , the wavelength inside the scatterer λ_a will also be much smaller than that in the background $\lambda_a \ll \lambda_b$. Thus, outside the scatterer the field will be a function of $k_b = \omega/c_b$, which is a slow oscillating function, while inside the scatterer the fields will be a function of $k_a = \omega/c_a$ and, therefore, it is a rapidly oscillating function. As we are in the low frequency limit we expect the medium behave as a homogeneous effective medium with constant parameters, but, due to the fields inside the scatterer, we will find that our effective medium has frequency-dependent parameters.

Next section analyzes this effect rigorously and for several type of scatterers, showing how complex scatterers present a frequency behavior that, as will be seen in Section 4, can lead to metamaterials with negative constitutive parameters.

3. Acoustic Scatterers with Locally Negative Parameters

The wave equation for the pressure field in an inhomogeneous fluid is given by [35]

$$\nabla(\rho^{-1}(\mathbf{r})\nabla P(\mathbf{r})) + \frac{\omega^2}{B(\mathbf{r})}P(\mathbf{r}) = 0 \quad (1)$$

where $\rho(\mathbf{r})$ and $B(\mathbf{r})$ are the mass density and bulk modulus, respectively, of the fluid material. The problem considered here is reduced to points in the $x - y$ plane, that in polar coordinates are $\mathbf{r} = (r, \theta)$. We also assume that a radially symmetric scatterer of radius R_a with some inhomogeneous parameters $\rho(r)$ and $B(r)$ is embedded into a fluid background with acoustic parameters ρ_b and B_b .

This is a canonical scattering problem whose solution outside the scatterer is given in terms of Bessel and Hankel functions [35],

$$P(r, \theta; \omega) = \sum_q A_q^0 [J_q(k_b r) + T_q H_q(k_b r)] e^{iq\theta} \quad , \quad r > R_a \quad (2)$$

with $k_b^2 = \omega^2 \rho_b / B_b$. The coefficients A_q^0 are determined by the incident field, and the response of the scatterer is described by the matrix elements T_q . This matrix is obtained by solving the wave equation (1) inside the scatterer and applying boundary conditions at $r = R_a$, which are the continuity of the pressure field and that of the normal component of the particle velocity,

$$P(R_a^+) = P(R_a^-) \quad (3a)$$

$$\frac{1}{\rho_b} \partial_r P(R_a^+) = \frac{1}{\rho(R_a)} \partial_r P(R_a^-). \quad (3b)$$

Since the scatterer is radially symmetric, and the parameters ρ and B depend only on the radial coordinate, the field inside the scatterer is expressed in a Fourier series of the form

$$P(r, \theta; \omega) = \sum_q B_q(\omega) \psi_q(r; \omega) e^{iq\theta}, \quad (4)$$

where the eigenfunctions $\psi_q(r; \omega)$ are solutions of the radial part of (1) in cylindrical coordinates,

$$\frac{\rho(r)}{r} \partial_r \left(\frac{r}{\rho(r)} \partial_r \psi_q(r; \omega) \right) + \left(\omega^2 \frac{\rho(r)}{B(r)} - \frac{q^2}{r^2} \right) \psi_q(r; \omega) = 0. \quad (5)$$

From this equation, after applying the boundary conditions, the general form for the T matrix is obtained

$$T_q = -\frac{\chi_q J'_q(k_b R_a) - J_q(k_b R_a)}{\chi_q H'_q(k_b R_a) - H_q(k_b R_a)} \quad , \quad \chi_q = \frac{\rho(R_a)}{\rho_b} \frac{\psi_q(R_a; \omega)}{\partial_r \psi_q(R_a; \omega)} k_b \quad (6)$$

This T matrix allows to distinguish the contribution of the background from that of the scatterer. The background contribution is described by the Bessel and Hankel functions, while the scatterer contribution is described by the function χ_q . Standard multiple scattering homogenization theory is based on the asymptotic forms of all these functions to derive, by means of the monopolar and dipolar terms ($q = 0$ and $q = 1$), the

effective medium properties, as explained in [32, 33]. However, metamaterial behavior can appear, as it is demonstrated below, in the regime where only Bessel and Hankel functions of the background take their asymptotic forms.

Let us then consider that the arguments of Bessel and Hankel functions are small ($k_b R_a \ll 1$), and use their asymptotic forms [36]. The monopolar component of the T matrix is

$$T_0 \approx \frac{i\pi R_a^2 k_b^2}{4} \frac{1 + \frac{1}{2} k_b R_a \chi_0}{\frac{k_b^2 R_a^2}{2} \ln k_b R_a - \frac{1}{2} k_b R_a \chi_0}, \quad (7)$$

where the logarithmic term in the denominator is obviously equal to zero in the low frequency limit. However it cannot be neglected when dealing with metamaterials. This term has been omitted in most of preceding works about acoustic and electromagnetic metamaterials, but it contributes considerably to their effective parameters.

Equivalently, the dipolar component of the T matrix is

$$T_1 \approx \frac{i\pi R_a^2}{4} \frac{\chi_1/k_b R_a - 1}{\chi_1/k_b R_a + 1} k_b^2. \quad (8)$$

Since we expect this scatterer behaves as a homogeneous scatterer with acoustic parameters ρ_a and B_a , the matrix elements should have the form

$$T_0 \approx \frac{i\pi R_a^2 k_b^2}{4} \left[\frac{B_b}{B_a} - 1 \right] \quad (9a)$$

$$T_1 \approx \frac{i\pi R_a^2}{4} \frac{\rho_a - \rho_b}{\rho_a + \rho_b} k_b^2. \quad (9b)$$

The comparison of these expressions with those given by equations (7) and (8), permits to identify the frequency-dependent bulk modulus and density as

$$B_a(\omega)/B_b = \frac{k_b^2 R_a^2}{2} \ln k_b R_a - \frac{1}{2} k_b R_a \chi_0 \quad (10a)$$

$$\rho_a(\omega)/\rho_b = \chi_1/k_b R_a \quad (10b)$$

Note that $B_a(\omega)$ and $\rho_a(\omega)$ are obtained from the fact that, after a scattering process (in the long wavelength limit) we expect to extract the parameters of a homogeneous fluid-like cylinder. These parameters are functions mass density at the surface of the scatterer, $\rho(r = R_a)$, but they also depend on the field and its derivative at the surface, that is, of $\psi_q(r = R_a)$ and $\partial_r \psi_q(r = R_a)$, respectively. These quantities are frequency-dependent and, consequently, they are the responsible of the frequency-dependence of parameters $B_a(\omega)$ and $\rho_a(\omega)$.

In Appendix A we study three types of scatterers giving negative parameters at very narrow frequency regions. The first one is a homogenous fluid like cylinder such that $c_a \ll c_b$, condition that grants $k_b \ll k_a$. The second is also a homogeneous fluid-like cylinder but now with cylindrical anisotropy. These type of cylinders have already been studied for cloaking devices, radial wave crystals [37] and hyperlenses [16] and here we will see how they can be used to tune the resonance of the dynamical mass density of a metamaterial. Finally the third example shows that fluid-like shells can work as

Helmholtz resonators, obtaining from them negative bulk modulus, but also they can work as metamaterials with negative mass density.

However, let us point out that we report here only the simpler examples of scatterers to realize acoustic metamaterials. Obviously more complex scatterers could be used to improve the frequency response, but a full analysis of this type is beyond the scope of the present work.

4. Multiple Scattering of Acoustic Waves in the Quasi-Static Limit

If we have a cluster of cylindrical scatterers defined by some ρ_a and B_a , we expect that, in the low frequency limit (that is, for wavelengths larger than the typical scatterer distance), they behave as a homogeneous medium with effective parameters ρ_{eff} and B_{eff} .

The comparison among the scattering properties of the cluster and the effective scatterer is used to obtain the effective parameters for the case of low filling fractions [43]. In [32, 33, 34] such scattering formulation was generalized and all the multiple scattering interactions between the scatterers were included, dealing to more general expressions which also include the possibility of having anisotropy in the mass density of the effective medium.

In the following subsections the results in [32, 33, 34] are generalized to the case of metamaterials with frequency-dependent parameters. It is assumed that we can replace ρ_a and B_a by their corresponding frequency dependent values $\rho_a(\omega)$ and $B_a(\omega)$. It will be shown that this method is self-consistent and, therefore, it defines the correct way to explain acoustic metamaterials.

4.1. Multipolar Interactions: The Δ Factor

Let us consider that a cluster of scatterers are periodically arranged in the space. In the low frequency limit such a cluster behaves like an effective fluid-like medium with effective parameters given by [43]

$$\frac{1}{B_{eff}(\omega)} = \frac{1-f}{B_b} + \frac{f}{B_a(\omega)} \quad (11a)$$

$$\rho_{eff}(\omega) = \frac{\rho_a(\omega)(1+f) + \rho_b(1-f)}{\rho_a(\omega)(1-f) + \rho_b(1+f)} \rho_b. \quad (11b)$$

In the above expressions the dependence on the frequency has been added under the assumption that we can use for a scatterer the frequency-dependent parameters defined by equations (10a) and (10b).

While equation (11a) is valid for all filling fractions, equation (11b) is valid for diluted clusters only (i.e. low filling fractions). In [33] and [34] the expressions for the effective density were generalized for the case of high filling fractions, and all the multiple scattering terms were introduced in Equation (11b) by means of the so called

Figure 1. Frequency-dependent bulk modulus for a medium composed of air cylinders in a water background.

Δ factor, leading to

$$\rho_{eff}(\omega) = \frac{\rho_a(\omega)(\Delta + f) + \rho_b(\Delta - f)}{\rho_a(\omega)(\Delta - f) + \rho_b(\Delta + f)} \rho_b, \quad (12)$$

The Δ factor represents a multipolar correction to the effective density expression, and it includes all the multiple scattering terms between all the cylinders in a cluster or in an infinite lattice. Such a factor contains also information about the density of the cylinders forming the cluster, therefore if we want to define a frequency dependent $\Delta = \Delta(\omega)$ factor and generalize the frequency dependent parameters to all filling fractions, the frequency-dependent mass density must also be included there. However this inclusion must be made carefully.

However this factor is only important for very high filling fractions and also for very strong scatterers [32, 33], so we consider that such correction out of the scope of the present work; it only adds complexity to the calculation of effective parameters.

4.2. Anisotropic Metamaterials

Expressions in previous section have not taken into account the possibility of having anisotropy; that is, they did not consider the lattice symmetry in which cylinders are

Figure 2. Imaginary component of the effective speed of sound in a medium composed of air cylinders in a water background.

ordered in the cluster. In [34] expressions for the anisotropic mass density were found in terms of the cylinder's parameters and the lattice geometry. It can be shown that, even neglecting the multiple scattering terms, we still have anisotropy for non-symmetric lattices, having the following expressions for the effective mass density tensor (the subindex “eff” has been omitted for clarity)

$$\rho_{xx}^{-1}(\omega) = \frac{1 - f^2\eta^2(\omega)(A + 1)^2}{1 + 2f\eta(\omega) + f^2\eta^2(\omega)(1 - A^2)}, \quad (13a)$$

$$\rho_{yy}^{-1}(\omega) = \frac{1 - f^2\eta^2(\omega)(A - 1)^2}{1 + 2f\eta(\omega) + f^2\eta^2(\omega)(1 - A^2)}, \quad (13b)$$

where A is the anisotropy factor explained in Appendix and $\eta(\omega)$ is defined as

$$\eta(\omega) = \frac{\rho_a(\omega) - \rho_b}{\rho_a(\omega) + \rho_b} \quad (14)$$

It has been assumed that the lattice is oriented along the main axis and, therefore, the tensor has been previously diagonalized.

The expression for the effective bulk modulus remains the same as in the previous section, so that we can obtain the effective sound speed tensor as [34]

$$c_{ij}^2 = \rho_{ij}^{-1} B_{eff} \quad (15)$$

Note that the anisotropic mass density tensor can have both components with the same sign (negative or positive) or each component with a different sign. However, as the refractive index is the square root of those terms, if we want a negative refractive index we need a negative bulk modulus too, therefore we can have only or both components

Figure 3. Table summarizing the different propagation behavior of acoustic metamaterials.

Figure 4. Effective parameters, normalized to those of the background, as a function of frequency of a rectangular lattice of fluid-like cylinders with $b = 2a$. The density, bulk modulus and radius of the cylinders are $\rho_a = 0.5\rho_b$, $B_a = 0.02B_b$ and $R_a = 0.49a$, respectively. We see two narrow multipolar resonances between $a/\lambda = 0.3$ and 0.35 , however, in that limit the homogenization hypothesis is not good. The thin horizontal dotted line is a guide for the eye.

Figure 5. Real part of the effective speed of sound tensor of the medium described in Fig. 4 as a function of frequency. Note that we can have both components positive, or one negative and the other one imaginary (real part equal to zero in this plot). It is impossible have both components negative or one negative and the other one positive (see text). The horizontal dotted line is a guide for the eye.

of the refractive index negative, or one negative and the other one imaginary (no wave propagation). In other words, we can have both an elliptical dispersion relation or a hyperbolic one. In Figure 4 the anisotropic mass density and the scalar bulk modulus have been plotted, and in Figure 5 the corresponding components of the sound speed tensor are shown.

5. Application to Electromagnetic Metamaterials

The vectorial nature of electromagnetic (EM) waves makes the problem more complex, but in 2D the EM field can be decomposed into TE and TM modes, leading to the same wave equation as for scalar acoustic waves. Now P in equation (1) and (5) is the z component of the electric (magnetic) field for TE (TM) modes, and $(\rho, B) = (\mu, \varepsilon^{-1})$ for TE modes and $(\rho, B) = (\varepsilon, \mu^{-1})$ for TM modes.

Although both problems are mathematically equivalent, physically they are quite different. The numerical values and ranges of the material parameters ρ, B and μ, ε are not the same in both fields, therefore it is worth to study them apart each other.

Thus, for TE modes

$$\frac{\varepsilon_b}{\varepsilon_a^{TE}(\omega)} = \frac{k_b^2 R_a^2}{2} \ln k_b R_a + \frac{k_a R_a}{2} \frac{J_0(k_a R_a)}{J_1(k_a R_a)} \frac{\varepsilon_b}{\varepsilon_a} \quad (16a)$$

$$\frac{\mu_a^{TE}(\omega)}{\mu_b} = \frac{1}{k_a R_a} \frac{J_1(k_a R_a)}{J_1'(k_a R_a)} \frac{\mu_a}{\mu_b} \quad (16b)$$

For TM modes,

$$\frac{\mu_b}{\mu_a^{TM}(\omega)} = \frac{k_b^2 R_a^2}{2} \ln k_b R_a + \frac{k_a R_a}{2} \frac{J_0(k_a R_a)}{J_1(k_a R_a)} \frac{\mu_b}{\mu_a} \quad (17a)$$

$$\frac{\varepsilon_a^{TM}(\omega)}{\varepsilon_b} = \frac{1}{k_a R_a} \frac{J_1(k_a R_a)}{J_1'(k_a R_a)} \frac{\varepsilon_a}{\varepsilon_b} \quad (17b)$$

These equations show that, even when the cylinders are non-magnetic ($\mu_a = \mu_b = \mu_0$), we can have a magnetic response for a given frequency range. This is a well known phenomenon called mesomagnetism[44, 45].

Also these equations show that, as a function of frequency, the same cylinders behave in a different way for TE or TM modes, presenting different constitutive parameters that are equal when $\omega \rightarrow 0$, that is

$$\lim_{\omega \rightarrow 0} \varepsilon_a^{TE}(\omega) = \varepsilon_a^{TM}(\omega) = \varepsilon_a \quad (18)$$

$$\lim_{\omega \rightarrow 0} \mu_a^{TE}(\omega) = \mu_a^{TM}(\omega) = \mu_a \quad (19)$$

The effective medium made of a cluster of these scatterers will present different constitutive parameters for each of the polarizations, so that, applying the results of Section 4,

$$\varepsilon_{eff}^{TE}(\omega) = (1 - f)\varepsilon_b + (1 - f)\varepsilon^{TE}b \quad (20a)$$

$$\mu_{eff}^{TE}(\omega) = \frac{\mu_a^{TE}(\omega)(1 + f) + \mu_b(1 - f)}{\mu_a^{TE}(\omega)(1 - f) + \mu_b(1 + f)} \mu_b \quad (20b)$$

$$\mu_{eff}^{TM}(\omega) = (1 - f)\mu_b + (1 - f)\mu^{TM}b \quad (20c)$$

$$\varepsilon_{eff}^{TM}(\omega) = \frac{\varepsilon_a^{TM}(\omega)(1 + f) + \varepsilon_b(1 - f)}{\varepsilon_a^{TM}(\omega)(1 - f) + \varepsilon_b(1 + f)} \mu_b \quad (20d)$$

Figure 6 depicts a plot of the effective constitutive parameters for a square lattice of dielectric cylinders with $\varepsilon_a = 11\varepsilon_b$ and $R_a = 0.4a$. Note that although $\mu_a = \mu_b = \mu_0$ there is a strong magnetic resonance for both polarizations. However we see that the resonance of ε^{TM} is beyond the homogenization limit ($\lambda \lesssim 4a$), so that probably this resonance could not be observed.

Figure 7 shows the effective speed of light (relative to that of the background) for this system. Note that only the TE polarization presents negative speed of light (or negative refraction index). This phenomenon is due to the fact that the resonance of ε^{TM} is too far and too sharp to interact with that of μ^{TM} . If we still want to have negative refraction in the two polarizations we can use anisotropic cylinders, as we did for the acoustic case.

If the cylinder's permittivity is given by a tensor of the form

$$\hat{\varepsilon}_a = (\varepsilon_{ar}, \varepsilon_{a\theta}, \varepsilon_{az}) \quad (21)$$

Figure 6. Effective constitutive parameters, relative to those of the background, for a square lattice of dielectric cylinders such that $R_a = 0.4a$ and $\varepsilon_a = 11\varepsilon_b$. The resonances of the permeability and permittivity for the TM case are so sharp that they do not allow to the system to present negative refraction for that polarization. However the TE polarization is allowed because of the wide range of negative permittivity.

Figure 7. Effective speed of light normalized to that of the background for both TE and TM modes. As we can see, we cannot have negative refraction for the TM mode (see text). The regions where there is not solution for the speed of light corresponds to those in which ε and μ have different sign, so that the speed of sound is imaginary.

Figure 8. Frequency shift to the left of ε^{TM} due to anisotropy in the cylinder. As we see, there is a frequency region in which all the constitutive parameters for the two modes (TE and TM) are negative, so that we have a region of total negative refraction.

the expressions for the frequency-dependent constitutive parameters are now

$$\frac{\varepsilon_b}{\varepsilon_a^{TE}(\omega)} = \frac{k_b^2 R_a^2}{2} \ln k_b R_a + \frac{k_a^{TE} R_a}{2} \frac{J_0(k_a^{TE} R_a)}{J_1(k_a^{TE} R_a)} \frac{\varepsilon_b}{\varepsilon_{az}} \quad (22a)$$

$$\frac{\mu_a^{TE}(\omega)}{\mu_b} = \frac{1}{k_a^{TE} R_a} \frac{J_1(k_a^{TE} R_a)}{J'_1(k_a^{TE} R_a)} \frac{\mu_a}{\mu_b} \quad (22b)$$

$$\frac{\mu_b}{\mu_a^{TM}(\omega)} = \frac{k_b^2 R_a^2}{2} \ln k_b R_a + \frac{k_a^{TM} R_a}{2} \frac{J_0(k_a^{TM} R_a)}{J_1(k_a^{TM} R_a)} \frac{\mu_b}{\mu_a} \quad (22c)$$

$$\frac{\varepsilon_a^{TM}(\omega)}{\varepsilon_b} = \frac{1}{k_a^{TM} R_a} \frac{J_\gamma(k_a^{TM} R_a)}{J'_\gamma(k_a^{TM} R_a)} \frac{\varepsilon_{a\theta}}{\varepsilon_b} \quad (22d)$$

where $k_a^{TE} = \omega \sqrt{\varepsilon_{za} \mu_a}$, $k_a^{TM} = \omega \sqrt{\varepsilon_{a\theta} \mu_a}$ and $\gamma^2 = \varepsilon_{a\theta} / \varepsilon_{ar}$.

These expressions allow us to shift the resonance of ε^{TM} to lower values just by increasing the ε_{ar} , keeping the rest of the system unaltered, in the same way as we did for the acoustic case. Note that the equivalence is $\rho_r \rightarrow \varepsilon_\theta^{TM}$ and $\rho_\theta \rightarrow \varepsilon_r^{TM}$, that is we need to increase the value of ε_{ar} in the electromagnetic case. Thus, as we increase it, the anisotropy factor γ goes to 0, and the resonance moves to the left.

Figure 8 considers the same system as in figure 6 but with anisotropic cylinders, where $\varepsilon_r = 200\varepsilon_\theta$ and $\varepsilon_\theta = \varepsilon_z = 11\varepsilon_b$. Note how all the resonances keep their positions but ε^{TM} , which now moves to the left reaching μ^{TM} , leading therefore to an effective medium with negative refraction. The effective speed of light is depicted in Fig 9, where it is obvious now that both polarizations present negative refraction properties within

Figure 9. Effective speed of light for both TE and TM modes for the system of Fig. 9. It is clear that there is a region of total negative refraction, thanks to the frequency shift of ε^{TM} due to anisotropy (see text).

a similar frequency range.

6. Summary

In summary, a comprehensive multiple scattering formulation of acoustic metamaterials has been introduced. This formulation is based on a homogenization theory in the quasi-static limit, in which we allow the wave number in the background be arbitrarily small while the wave number in the scatterers remains finite.

In general, it is shown that ordered or disordered arrays of sound scatterers can behave, in the low frequency limit, like effective fluid-like materials with either positive or negative acoustic parameters, where these acoustic parameters are the scalar bulk modulus and the tensorial mass density. The behavior of the effective medium depends, among other properties, on the surface fields in the scatterer. Therefore, it is possible to improve or manage the frequency response of the effective medium with complex scatterers, like fluid-like shells or anisotropic fluid-like materials.

Examples of these scatterers have been analyzed, showing that they present negative effective parameters whenever the theory predicts them, verifying therefore the formulation presented. Also, it has been shown how these complex scatterers can be used to tune the effective parameters of the medium.

The theory developed for acoustic waves has been extended to electromagnetic waves in 2D, showing that equivalent type of scatterers can also be used in electromagnetism to tune the effective medium response.

Figure A1. Frequency-dependent acoustic parameters of a homogeneous cylinder with $B_a = 0.005B_b$ and $\rho_a = 0.5\rho_b$. The cylinder's radius is $R_a = 0.3a$, where a is the lattice constant.

In conclusion, the theory presented not only explains the metamaterial behavior found so far in the literature, but also gives the basis for improve this behavior with more complex scatterers.

Acknowledges

This work was partially supported by U.S. Office of Naval Research under grant No. N000140910554 and the Spanish Ministry of Science and Innovation under Contracts No. TEC2010-19751 and No. CSD2008-66 (CONSOLIDER Program). Daniel Torrent also acknowledges the contract provided by the program Campus de Excelencia Internacional 2010 UPV.

Appendix A. Examples of Scatterers with Local Negative Parameters

Appendix A.1. Homogeneous and Isotropic Scatterers

For a homogeneous scatterer with parameters ρ_a and B_a the field inside the scatterer is given in terms of Bessel functions. Therefore, after some algebra, the frequency

dependent parameters are

$$B_a(\omega)/B_b = \frac{k_b^2 R_a^2}{2} \ln k_b R_a + \frac{k_a R_a}{2} \frac{J_0(k_a R_a)}{J_1(k_a R_a)} \frac{B_a}{B_b}, \quad (1.1a)$$

$$\rho_a(\omega)/\rho_b = \frac{1}{k_a R_a} \frac{J_1(k_a R_a)}{J'_1(k_a R_a)} \frac{\rho_a}{\rho_b}, \quad (1.1b)$$

where $k_a = \omega \sqrt{\rho_a/B_a}$. For $k_a \rightarrow 0$ we recover the cylinder's parameters. Note that the metamaterial concept appears when $k_b \ll 1$ but k_a is not small enough. This condition occurs when $c_a \ll c_b$; i.e. when the wavelength in the background is several times larger than that inside the scatterer.

Figure A1 plots the frequency-dependent parameters described by Eqs. (10a) and (10b) for a soft scatterer with $B_a = 0.005B_b$ and $\rho_a = 0.5\rho_b$. These values give a sound speed of the scatterer $c_a = 0.1c_b$, which locates the resonances in the low frequency limit, as can be seen in the figure.

From (10b) it is deduced that the region of negativity for $\rho_a(\omega)$ is determined by the first zeros of $J_1(k_a R_a)$ and $J'_1(k_a R_a)$, which are $\alpha = 3.8317$ and $\alpha' = 1.8412$, respectively[36], and corresponds to frequencies ω_- such that

$$\frac{1.8412c_a}{R_a} < \omega_- < \frac{3.8317c_a}{R_a} \quad , \quad \Delta\omega_- \approx \frac{2c_a}{R_a} \quad (1.2)$$

Therefore, if we want to locate the frequency region in the low frequency limit, we have to decrease c_a (for fixed R_a). But, as a consequence, the bandwidth will decrease; that is, the resonance becomes sharper.

These type of scatterers are possible only in a dense background, like water, where we can get such a low bulk modulus and density. If we need to get metamaterials in an air or gas background another approach should be used instead.

Appendix A.2. Homogeneous and Anisotropic Scatterers

Sometimes it is not possible to obtain sound speed smaller than a certain value but we still want to decrease the frequency at which negative mass density appears. In this case, fluid-like cylinders with circular anisotropy [38] can be used to shift the resonance to lower frequencies.

Anisotropic cylinders are characterized by a scalar bulk modulus B_a and a tensorial mass density whose components are constant when referred to a cylindrical coordinate system, $\hat{\rho}_a = (\rho_r, \rho_\theta)$. In these cylinders the pressure field is described in terms of Bessel functions of real order γq , where $\gamma = \sqrt{\rho_r/\rho_\theta}$ is the anisotropy factor,

$$\psi_q(r, \omega) = J_{\gamma q}(k_a r) \quad (1.3)$$

Despite being anisotropic cylinders, the circular symmetry of these scatterers make them suitable for applying the theory developed in this work. Thus, when $q = 0$ the field distribution is the same than that of an isotropic cylinder (because $J_{\gamma q}$ for $q = 0$

Figure A2. Frequency-dependent dynamical mass density of an anisotropic fluid like cylinder for three different anisotropy ratios. The radial component of the sound speed tensor is $c_r = 0.2c_b$ in the three examples. We see that, as we increase the anisotropy ratio, the resonance is shifted to lower frequencies.

is J_0), therefore the frequency dependent bulk modulus will be also given by (1.1a). However (1.1b) now becomes

$$\rho_a(\omega)/\rho_b = \frac{1}{k_a R_a} \frac{J_\gamma(k_a R_a)}{J'_\gamma(k_a R_a)} \frac{\rho_a}{\rho_b} \quad (1.4)$$

and, as we let the anisotropy factor γ be smaller, the dipolar resonance $q = 1$ becomes closer to the monopolar one,

$$\lim_{\gamma \rightarrow 0} J_\gamma(k_a r) \approx J_0(k_a r) \quad (1.5)$$

decreasing therefore the resonant frequency.

In Fig. A2 the frequency-dependent mass density is plotted for three different ρ_r/ρ_θ ratios. Note how as we increase the value of ρ_θ (so that we decrease the anisotropy factor γ) the resonance of the density moves down in frequency.

These type of strongly anisotropic fluid-like cylinders have already been used by Li et al. [16] for building acoustic hyperlens and it has been recently characterized for different anisotropy ratios in [39], showing also the same frequency shift.

Appendix A.3. Fluid-Like Shells as Helmholtz Resonators

Let us assume now that we have a fluid like cylinder of radius R_a and parameters ρ_a and B_a . If this cylinder is enclosed by another of radius $R_b > R_a$ and parameters ρ_s and B_s we have a fluid-like shell. Obviously this is an idealization, because such a structure

Figure A3. Frequency-dependent bulk modulus of a water shell in air with $\rho_s = 1000\rho_b$ and $c_s = 4.4c_b$. The outer radius of the shell is $R_b = 0.3a$, and the plot shows three different radii R_a .

cannot be realized with common fluids. However, on the one hand, if the shell is an elastic material, this can be sometimes a good approximation and, on the other hand, if the fluids are “metafluids” [40, 41, 42], the structure can be easily fabricated.

These structures can work, as Helmholtz resonators [24, 26]. Here we give a more rigorous derivation of the resonance frequency.

After applying the correct boundary conditions, the impedance factor χ_q of a fluid-like shell is given by

$$\chi_q = \frac{\rho_s c_s}{\rho_b c_b} \frac{J_q(k_s R_b) + T_q^a Y_q(k_s R_b)}{J'_q(k_s R_b) + T_q^a Y'_q(k_s R_b)}, \quad (1.6)$$

where

$$T_q^a = -\frac{\chi_q^a J'_q(k_s R_a) - J_q(k_s R_a)}{\chi_q^a H'_q(k_s R_a) - H_q(k_s R_a)}, \quad (1.7)$$

and

$$\chi_q^a = \frac{\rho_a c_a}{\rho_s c_s} \frac{J_q(k_a R_a)}{J'_q(k_a R_a)}. \quad (1.8)$$

The parameters of the shell are ρ_s and c_s , while those of the enclosed cylinder are ρ_a and c_a .

For $q = 0$ and $k_s \rightarrow 0$ the inner T matrix T_0^a is equal to

$$T_0^a \approx \frac{\pi R_a^2 k_s^2}{4} \frac{\left[1 - \frac{B_s}{B_a}\right]}{\xi_a}, \quad (1.9)$$

where

$$\xi_a = 1 + \frac{B_s}{B_a} \frac{k_s^2 R_a^2}{2} \ln k_s R_a. \quad (1.10)$$

Thus the impedance factor of the shell can be approximated to

$$\chi_0 \approx -\frac{2}{k_b R_b} \frac{B_s}{B_b} \frac{1 + \frac{k_s^2 R_a^2}{2} \left[1 - \frac{B_s}{B_a}\right] \ln k_s R_b / \xi_a}{1 - \frac{R_a^2}{R_b^2} \left[1 - \frac{B_s}{B_a}\right] / \xi_a}, \quad (1.11)$$

which can be zero only for $B_s/B_a \gg 1$ and consequently

$$\xi_a = \frac{k_s^2 R_a^2}{2} \frac{B_s}{B_a} \ln k_s R_b \quad (1.12)$$

which defines a cut off frequency ω_c

$$\omega_c = \frac{c_a}{R_a} \sqrt{\frac{2\rho_a}{\rho_s \ln(R_b/R_a)}} \quad (1.13)$$

If the shell is soft, that is, if $B_s/B_a \ll 1$, a negative bulk modulus appears as in the homogeneous cylinder case, and the shell nature of the scatterer is not relevant. In that case, as $R_b \rightarrow R_a$ the impedance factor reduces to

$$\chi_0 \approx -\frac{2}{k_b R_b} \frac{B_s}{B_b} \frac{1}{1 - \frac{R_a^2}{R_b^2} \left[1 - \frac{B_s}{B_a}\right]} \approx -\frac{2}{k_b R_b} \frac{B_a}{B_b} \quad (1.14)$$

and the bulk modulus cannot be negative, however we will see now that, in that case, the density can be negative.

When $q = 1$ the inner T matrix T_1^a is approximated by

$$T_1^a \approx -\frac{\pi R_a^2 k_s^2}{4} \frac{\rho_a - \rho_s}{\rho_a + \rho_s} \quad (1.15)$$

the density becomes negative once the denominator of χ_1 cancels, that is, when

$$J_1'(k_s R_b) + T_1^a Y_1'(k_s R_b) \approx \frac{1}{2} \left[1 - \frac{k_s^2 R_b^2}{4} - \frac{R_a^2}{R_b^2} \frac{\rho_a - \rho_s}{\rho_a + \rho_s}\right] = 0. \quad (1.16)$$

This expression gives a cut off frequency for the negative density of

$$\omega_c^2 = \frac{4c_s^2}{R_b^2} \left[1 - \frac{R_a^2}{R_b^2} \frac{\rho_a - \rho_s}{\rho_a + \rho_s}\right], \quad (1.17)$$

where now we need that $\rho_s \ll \rho_a$.

Appendix B. Technical Details

Appendix B.1. The anisotropy factor A

A two dimensional periodic array of scatterers is defined by the lattice vectors \mathbf{a}_1 and \mathbf{a}_2 , so that the position \mathbf{R}_n of any scatterer in the lattice can be determined by two integers n_1 and n_2 such that

$$\mathbf{R}_n = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 \quad (2.1)$$

This lattice has also associated the reciprocal lattice vectors \mathbf{b}_1 and \mathbf{b}_2 such that

$$\mathbf{b}_i \cdot \mathbf{a}_j = 2\pi\delta_{ij} \quad , \quad i = 1, 2 \quad (2.2)$$

If we define the reciprocal lattice point $\mathbf{G}_h = (G_h, \theta_h)$ as

$$\mathbf{G}_h = h_1\mathbf{b}_1 + h_2\mathbf{b}_2 \quad (2.3)$$

the anisotropy factor A can be found in [34] and is given by

$$A = 48 \sum_{h_1, h_2 \neq 0} \frac{J_3(G_h R_{min})}{G_h^3 R_{min}^3} e^{-2i\theta_h} \quad (2.4)$$

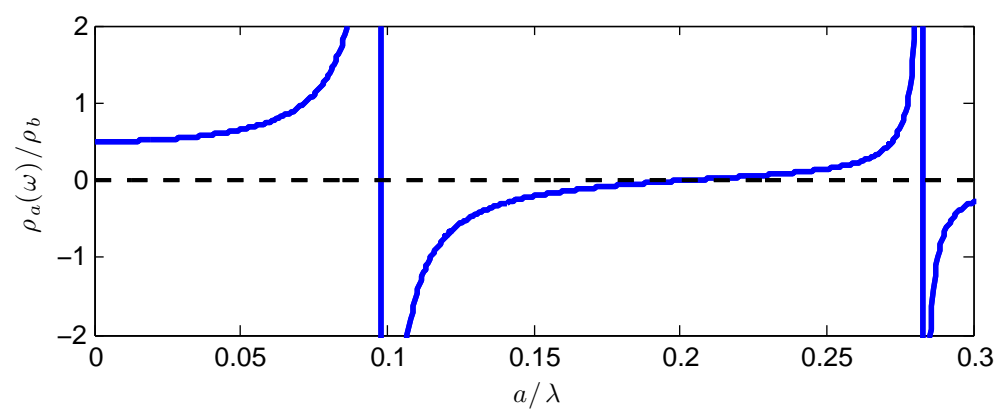
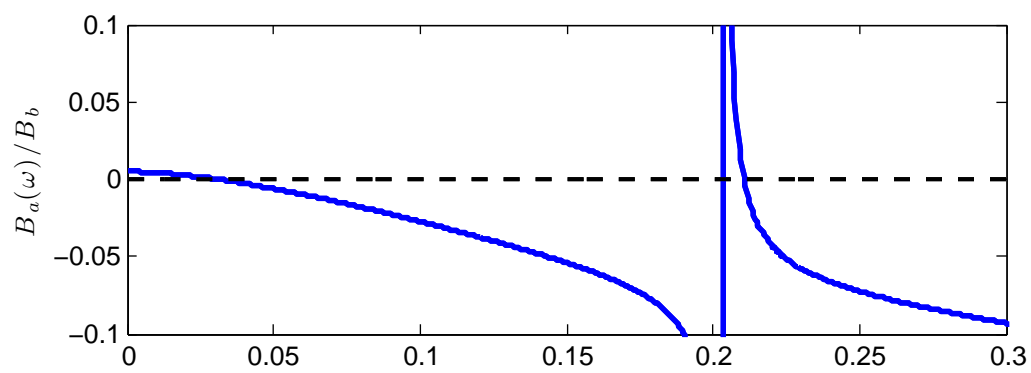
where $J_3(\cdot)$ is the third order Bessel function and R_{min} is the smaller of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_1 + \mathbf{b}_2$. Factor A can be made always real by properly choosing a coordinate system in which the tensors be diagonal.

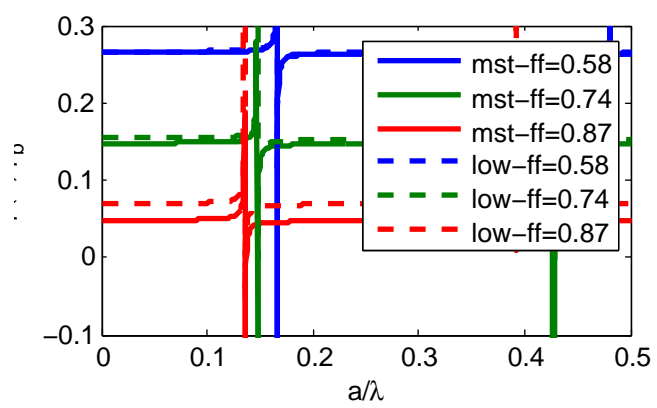
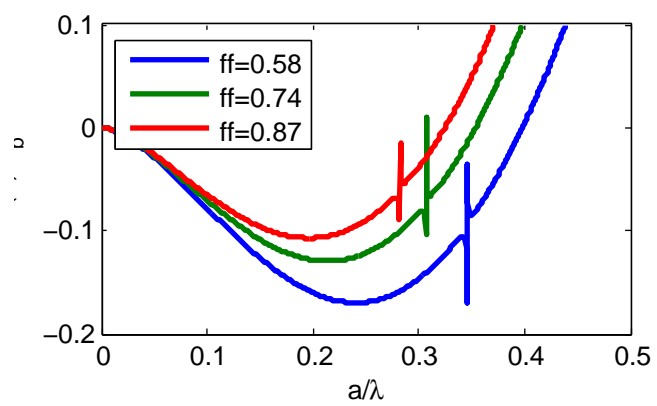
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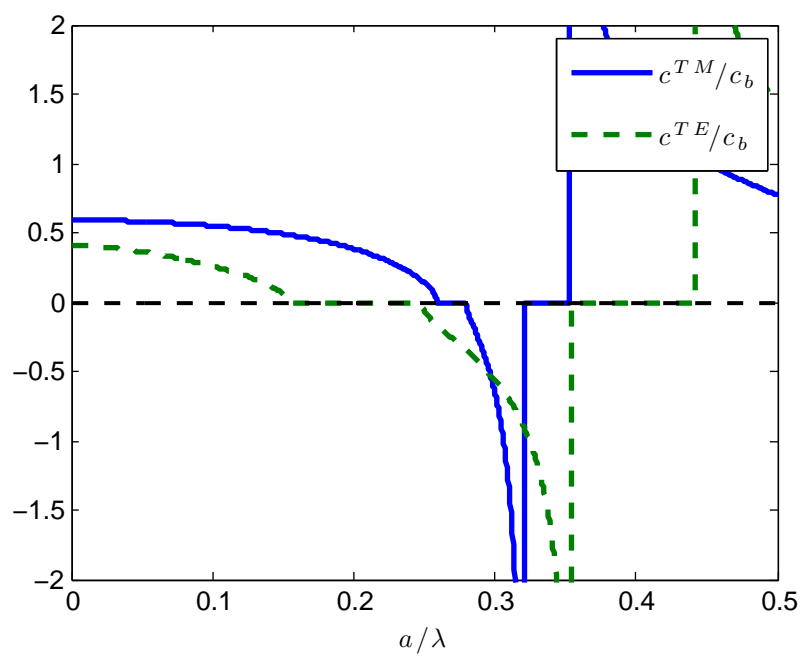
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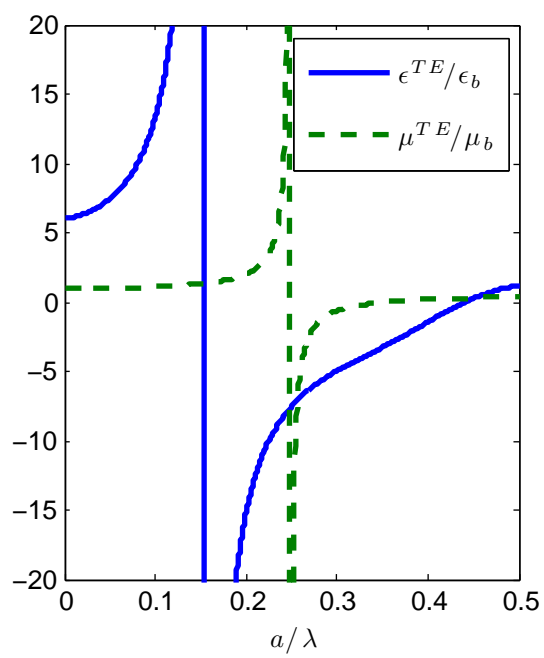
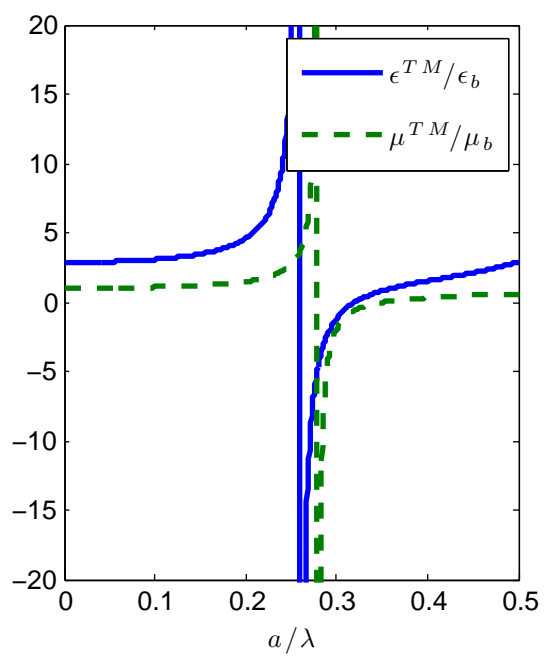
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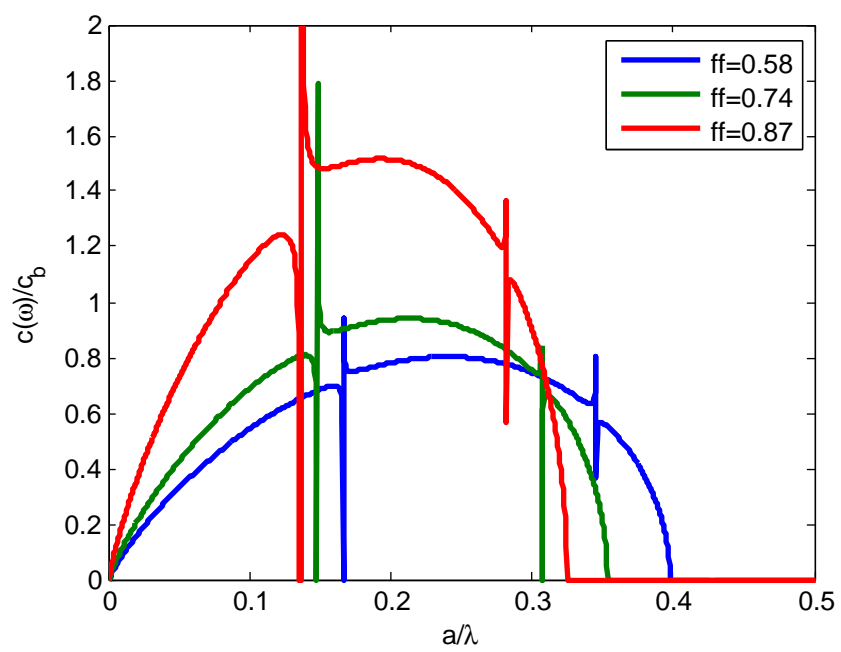
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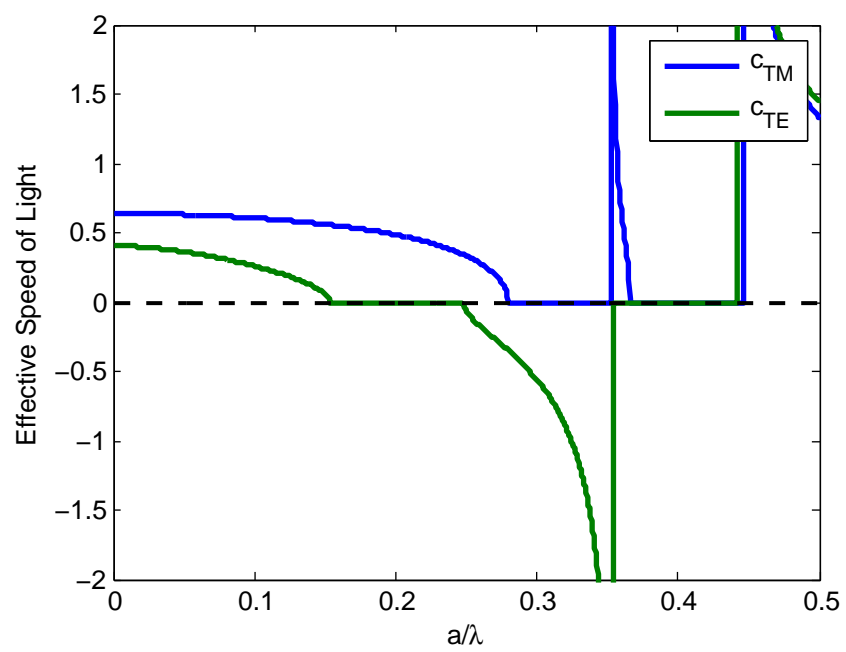


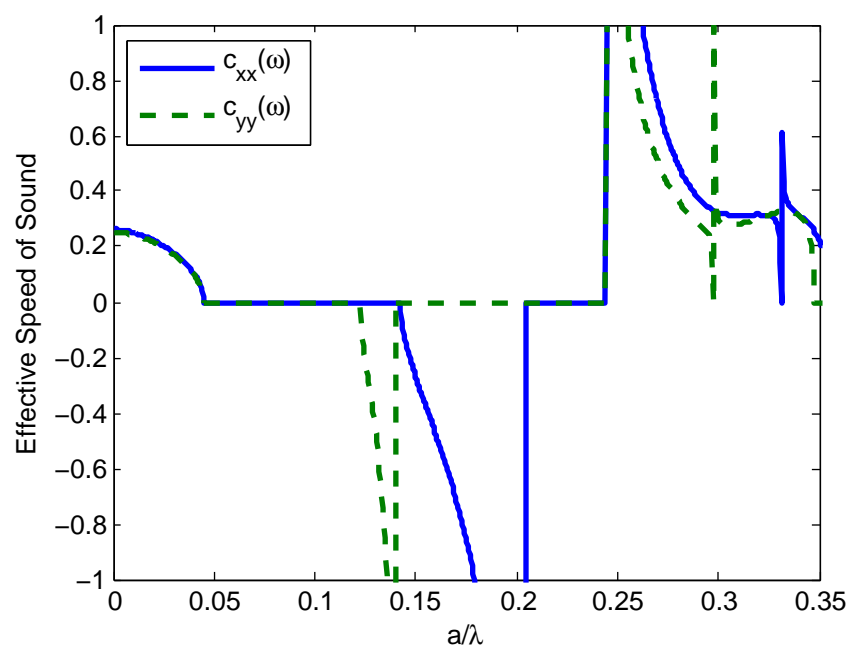


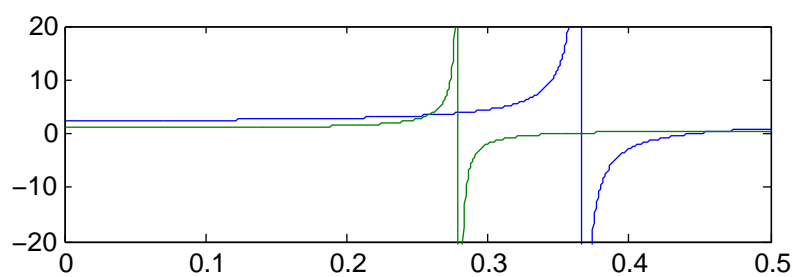
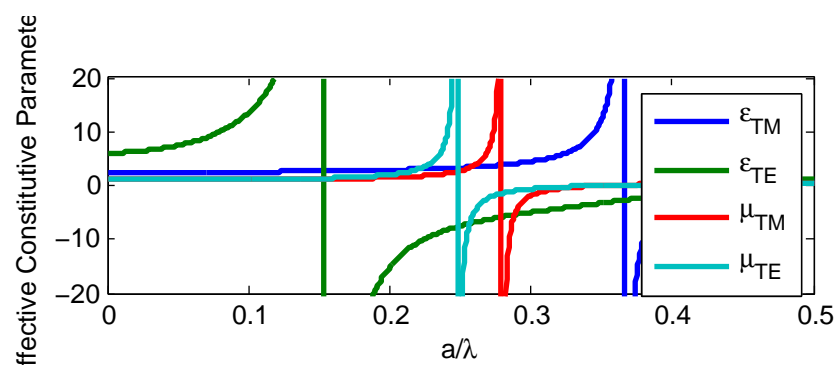


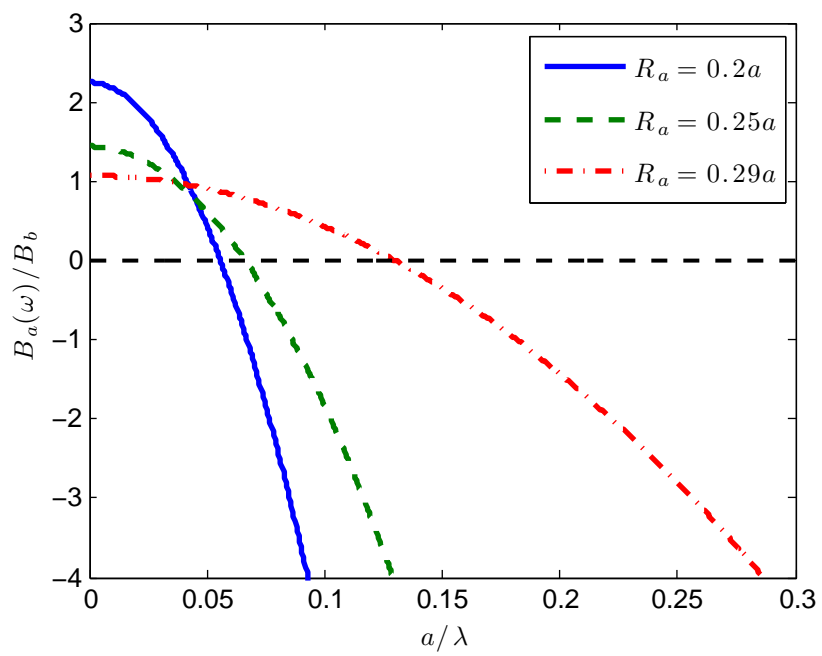


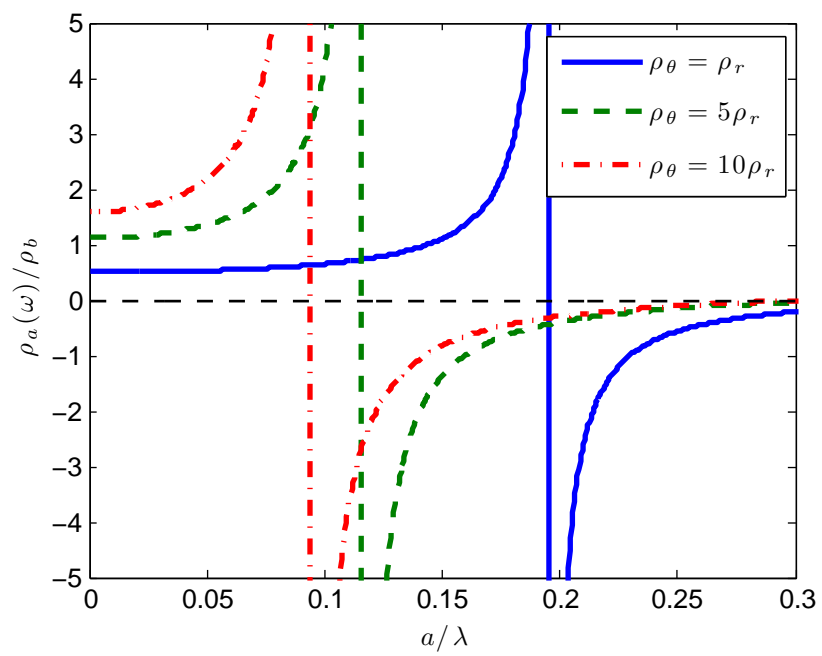


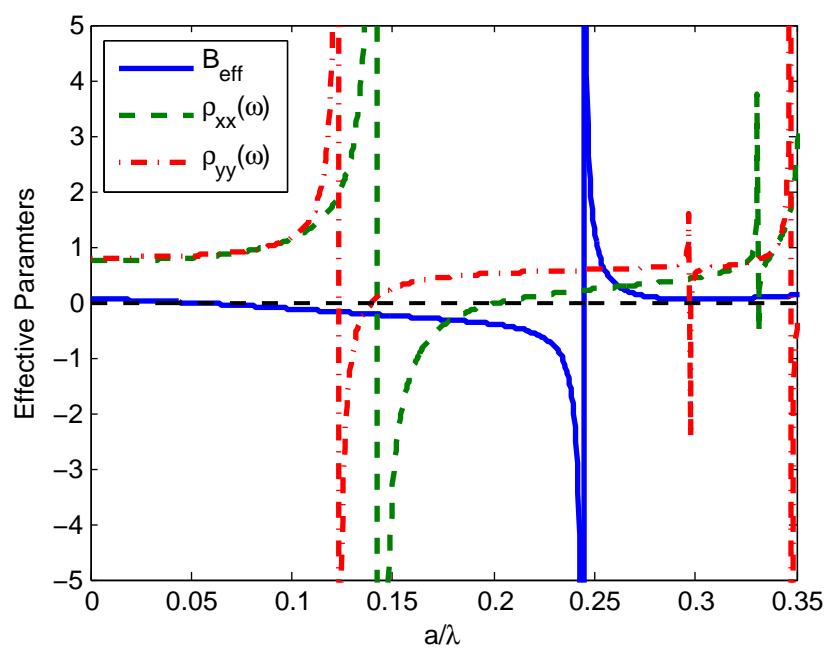












$B > 0$ $B < 0$	$\rho_{xx} < 0$	$\rho_{xx} > 0$
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$\rho_{yy} < 0$	<div>No Propagation</div> <div>Elliptic Negative</div>	<div>Hyperbolic Positive</div> <div>Hyperbolic Negative</div>